

NOTE

HADAMARD MATRICES OF ORDER 764 EXIST

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Received March 11, 2007

Two Hadamard matrices of order 764 of Goethals–Seidel type are constructed.

Recall that a Hadamard matrix of order m is a $\{\pm 1\}$ -matrix A of size $m \times m$ such that $AA^T = mI_m$, where T denotes the transpose and I_m the identity matrix. We refer the reader to one of [2, 4] for the survey of known results about Hadamard matrices.

In our previous note [1], written about 13 years ago, we listed 17 integers $n \leq 500$ for which no Hadamard matrix of order $4n$ was known at that time. Two of these integers were removed in that note and the smallest one, $n = 107$, was removed recently by Kharaghani and Tayfeh-Rezaie [3]. Among the remaining 14 integers n only four are less than 1000. The problem of existence of Hadamard matrices of these four orders, namely 668, 716, 764 and 892, has been singled out as Research Problem 7 in the recent book [2] by Kathy Horadam. In this note we shall remove the integer 764 from the mentioned list by constructing two examples of Hadamard matrices of Goethals–Seidel type of that order. (We have constructed a bunch of examples but we shall present only two of them.) Consequently, the revised list now consists of 13 integers:

167, 179, 223, 251, 283, 311, 347, 359, 419, 443, 479, 487, 491;

all of them primes congruent to 3 (mod 4).

Mathematics Subject Classification (2000): 05B20, 05B30

* The author was supported by an NSERC Discovery Grant.

For the remainder of this note we set $n = 191$. Let G be the multiplicative group of non-zero residue classes modulo the prime $n = 191$, a cyclic group of order $n - 1 = 190$, and let $H = \langle 39 \rangle = \{1, 39, 184, 109, 49\}$ be its subgroup of order 5. We choose the enumeration of the 38 cosets α_i , $0 \leq i \leq 37$, of H in G so that $\alpha_{2i+1} = -1 \cdot \alpha_{2i}$ for $0 \leq i \leq 18$ and

$$\begin{aligned} \alpha_0 &= H, & \alpha_2 &= 2H, & \alpha_4 &= 3H, & \alpha_6 &= 4H, & \alpha_8 &= 6H, \\ \alpha_{10} &= 8H, & \alpha_{12} &= 9H, & \alpha_{14} &= 11H, & \alpha_{16} &= 12H, & \alpha_{18} &= 13H, \\ \alpha_{20} &= 16H, & \alpha_{22} &= 17H, & \alpha_{24} &= 18H, & \alpha_{26} &= 19H, & \alpha_{28} &= 22H, \\ \alpha_{30} &= 32H, & \alpha_{32} &= 36H, & \alpha_{34} &= 38H, & \alpha_{36} &= 41H. \end{aligned}$$

Define four index sets:

$$\begin{aligned} J_1 &= \{1, 7, 9, 10, 11, 13, 17, 18, 25, 26, 30, 31, 33, 34, 35, 36, 37\}, \\ J_2 &= \{1, 4, 7, 9, 11, 12, 13, 14, 19, 21, 22, 23, 24, 25, 26, 29, 36, 37\}, \\ J_3 &= \{0, 3, 4, 5, 7, 8, 9, 16, 17, 19, 24, 25, 29, 30, 31, 33, 35, 37\}, \\ J_4 &= \{1, 3, 4, 5, 8, 11, 14, 18, 19, 20, 21, 23, 24, 25, 28, 29, 30, 32, 34, 35\} \end{aligned}$$

and introduce the following four sets of integers modulo 191:

$$S_k = \bigcup_{i \in J_k} \alpha_i, \quad k = 1, 2, 3, 4.$$

Their cardinals $n_k = |S_k| = 5|J_k|$ are:

$$n_1 = 85, \quad n_2 = n_3 = 90, \quad n_4 = 100$$

and we set

$$\lambda = n_1 + n_2 + n_3 + n_4 - n = 174.$$

For $r \in \{1, 2, \dots, 190\}$ let $\lambda_k(r)$ denote the number of solutions of the congruence $i - j \equiv r \pmod{191}$ with $\{i, j\} \subseteq S_k$. It is easy to verify (by using a computer) that

$$\lambda_1(r) + \lambda_2(r) + \lambda_3(r) + \lambda_4(r) = \lambda$$

is valid for all such r . Hence the sets S_1, S_2, S_3, S_4 are supplementary difference sets (SDS), with associated decomposition

$$\begin{aligned} 4n = 764 &= 9^2 + 11^2 + 11^2 + 21^2 \\ &= \sum_{k=1}^4 (n - 2n_k)^2. \end{aligned}$$

Let A_k be the $n \times n$ circulant matrix with first row

$$a_{k,0}, a_{k,1}, \dots, a_{k,n-1}$$

where $a_{k,j} = -1$ if $j \in S_k$ and $a_{k,j} = 1$ otherwise. These $\{\pm 1\}$ -matrices satisfy the identity

$$\sum_{k=1}^4 A_k A_k^T = 4nI_n.$$

One can now plug in the matrices A_k into the Goethals–Seidel array to obtain a Hadamard matrix of order $4n = 764$.

Our second example is constructed in the same way by using the index sets:

$$\begin{aligned} J_1 &= \{0, 1, 6, 8, 9, 11, 12, 16, 18, 20, 21, 23, 28, 31, 33, 36, 37\}, \\ J_2 &= \{0, 1, 3, 4, 10, 12, 13, 17, 20, 22, 24, 31, 32, 33, 34, 35, 36, 37\}, \\ J_3 &= \{4, 8, 9, 10, 12, 13, 14, 16, 17, 20, 21, 24, 26, 27, 29, 31, 32, 34\}, \\ J_4 &= \{1, 7, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 23, 25, 28, 29, 32, 33, 34, 37\}. \end{aligned}$$

The two solutions are not equivalent in the sense that the two SDS's are not equivalent. (For the definition of equivalence for SDS's see our note [1].)

References

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